Lesson 1: Radians, Angular Displacement, & Angular Velocity

We've already looked at regular **linear motion** (aka translational motion) as a way of describing the basic motion of an object in a straight line in terms of things like displacement, velocity, and acceleration.

We will now look at **rotational motion**, objects that are spinning.

- We will assume that the object is a **rigid body**. This just means that it keeps it shape, even as it spins. Imagine a wheel on a bicycle; even as it turns we expect it to keep the same radius the entire time.
- In reality, objects may change their shape slightly while spinning, but we will ignore this since it is usually a small, unimportant change.

Radians

In order to handle what we are about to look at, it is necessary to first understand **radians**.

- This may seem at first like a math thing, or not worth the trouble. Although we can describe rotational motion using degrees, it's just not the way that it is most often done.

Radians
https://tinyurl.com/3y4yv6d8

The part we should be focusing on is how we will measure the distance along the curve of a circle for a certain angle. We call this distance the **arc length**, and show it in formulas as **s**.

- To get an idea of this, imagine a circle as shown below. We have set a **reference line** on the positive x-axis and will start our measurements from there. This means the **reference line** has an angle of zero.
- Let's say we want to measure an angle of exactly **1 radian** going around counterclockwise from the **reference line** This means that the distance measured along the circumference of the circle must have an **arc length** of exactly **one radius**.

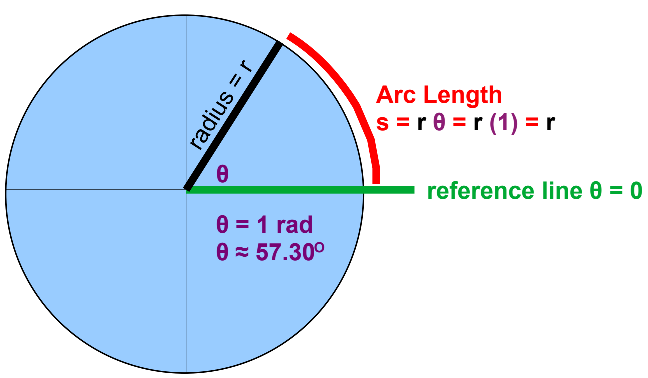


Figure 1: An angle of one radian has an arc length, s, equal to the radius, r.

$$\boxed{\theta = \frac{s}{r}}$$

θ = angle (rad)
s = arc length*
r = radius*

** any unit, as long as the same*

Notes:
Rads is dimensionless (has no real units) since it is the ratio between two lengths.

It may help you to remember from math and studying the unit circle that…
C = 2πr = 1 revolution = 360° = 2π rad

The great part about this system is that even if the circle is bigger or smaller, if the angle is **1 rad** then the **arc length** is equal to the **radius**.

- For any fraction or multiple of radians, we can measure any **arc length** we want.

Example 1: Owls are known for their amazing eyesight which allows them to spot very small prey like mice or voles from long distances. They can discern between different objects down to an angle as small as 2.0e-4 rad. **Determine** how small its prey can be if an owl is flying 150 m above a field.

Although it does not give us an exact answer, the arc length will be approximately the same size as the smallest prey the owl can spot. The radius is the height the owl is flying at.

$$\theta = \frac{s}{r}$$
$$s = r\theta$$
$$s = 150\,m(2.0\text{e-}4\,rad)$$
$$s = 0.030\,m = 3.0\,cm$$

Angular Position & Angular Displacement

We can define the **angular position**, **θ**, it terms of radians if we look at different positions of a point on the rigid body as we turn about the circle. Note, that by tradition…

- Going counterclockwise around the **axis of rotation** (the centre) is considered to be the positive direction, so $\theta > 0$
- Going clockwise around the **axis of rotation** is considered to be the negative direction, so $\theta < 0$

Imagine that we have a point on the spinning object that starts at the reference line (*Figure 3*).

- A moment later we see it has moved counterclockwise on the body (that's going in the positive direction).
- For the **arc length**, **s**, that it has moved, we would define its **angular position** as **θ** measured in **radians**.

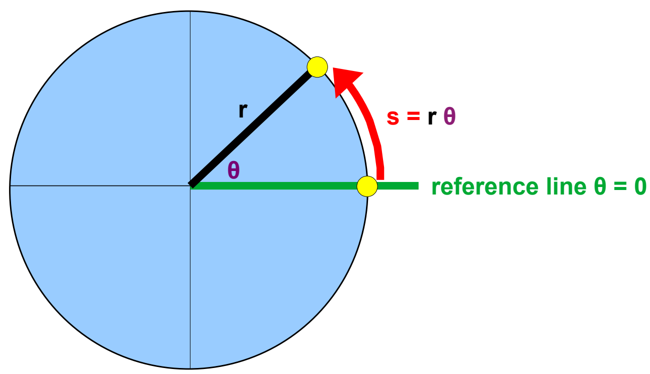


Figure 2: The angular position of this yellow dot is a positive radian measure.

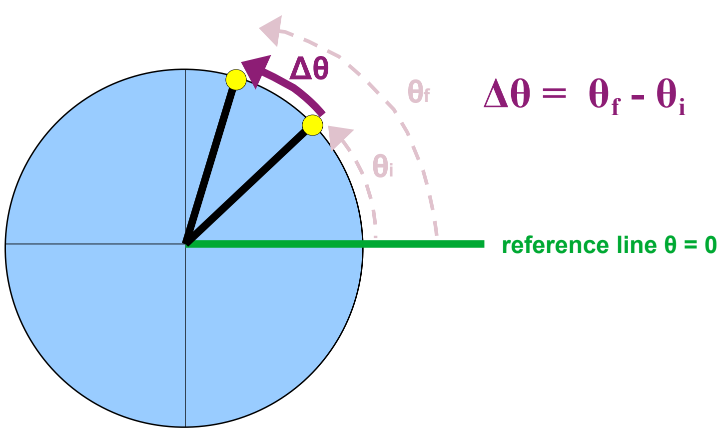


Figure 3: Angular displacement is the difference between two angular positions.

If we want to continue to watch the point as it moves around in the circle, we can measure its **angular displacement, Δθ** (*Figure 4*).

- Like any displacement, this means we need to measure the difference between two positions, where it started, θ_i , and where it ends, θ_f .

Angular Velocity

Now that we have a way of defining how far something has moved, the **angular displacement**, we can measure the **average angular velocity**.

- When we were doing regular, linear velocity, we just used the basic formula…

$$v = \frac{\Delta d}{\Delta t}$$

- We can take a similar approach now, since the basic idea behind the calculation is still a rate of change formula, where we need to measure how far something has moved in an interval of time.

$$\boxed{\omega = \frac{\Delta \theta}{\Delta t}}$$

Note:
Since rads is unitless, you may see the units for angular velocity given as 1/s, which is the same as frequency. More on this soon!

ω = "omega" = average angular velocity (rads/s)
$\Delta\theta$ = angular displacement (rads)
Δt = time (s)

Measuring angular velocity is convenient in that *all points* on the spinning object have the *same* angular velocity.

- This is true since it is based on radians, not any individual length measured on the circle.
- Since *radians is the ratio between arc length and radius*, $\theta = \frac{s}{r}$, as we get further away from the centre of rotation (radius gets bigger), the arc length proportionately also gets bigger, ***but the angle (and the angular displacement) is still the same***.
- Because the angular velocity is only directly related to the angular displacement, the angular velocity stays the same.

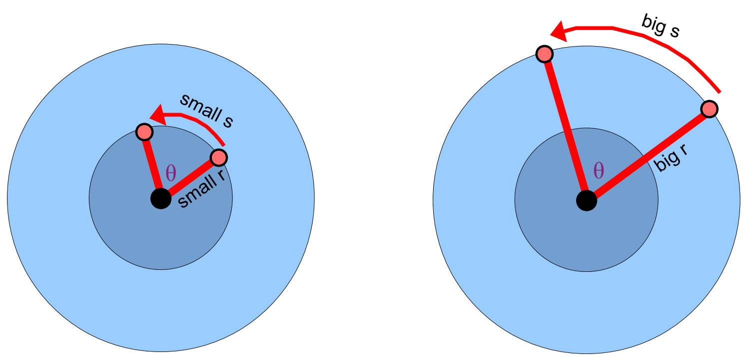


Figure 4: The angular velocity is the same for any point on a spinning rigid body.

We can start to do some manipulations with our angular velocity formula to relate it to some other ideas.

- We know that the time it takes to move through exactly one complete revolution is the **period**, ***T***, measured in seconds.
- We also know that one complete revolution around in a circle is when the **angular displacement**, ***Δθ***, is equal to 2π.
- So, if we measure the motion of a point on the circle as it moves all the way around once…

$$\omega = \frac{\Delta\theta}{\Delta t}$$ where we can substitute in $\Delta\theta = 2\pi$ and $\Delta t = T$

$$\boxed{\omega = \frac{2\pi}{T}}$$

ω = angular velocity (rads/s)
T = period (s)

Manipulating the formula to solve for T gives us...

$$T = \frac{2\pi}{\omega}$$ which we can combine with $$T = \frac{1}{f}$$

$$\frac{2\pi}{\omega} = \frac{1}{f}$$

$$\boxed{\omega = 2\pi f}$$

ω = angular velocity (rads/s)
f = frequency (1/s or Hz)

Note:
Measureing frequency in Hertz is really just cycles/second, just like angular velocity is rads/second. Both really measure "how much" per second!

Example 2: On old record players, one of the settings would spin the records at 45 rpm. **Determine** the angular velocity of the record.

To solve this, you do need to recall that we must first convert frequencies in rpm (revolutions per **minute**) into Hz (revolutions per **second**).

$$45\frac{rev}{min}\left(\frac{1\,min}{60\,sec}\right) = 0.75\,Hz$$

$$\omega = 2\pi f$$
$$\omega = 2(3.14)(0.75)$$
$$\omega = 4.71 = 4.7\,rads/s$$

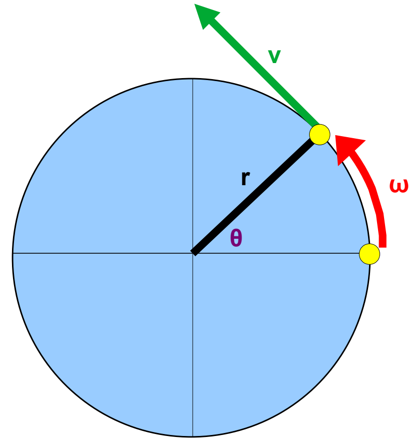
Example 3: CD players spin with an angular velocity of 22.0 rad/s. **Determine** the frequency of a CD as it spins, measured in both Hertz and rpm.

$$\omega = 2\pi f$$
$$f = \frac{\omega}{2\pi}$$
$$f = \frac{22.0}{2(3.14)}$$
$$f = 3.50\,Hz$$

$$3.50\frac{rev}{s}\left(\frac{60\,s}{1\,min}\right) = 210\,rpm$$

The CD spins at 3.50 Hz, which is the same as 210 rpm.

Relating Angular Velocity to Linear Velocity



Any point on a rotating rigid body must also have an instantaneous **linear velocity**.

- This instantaneous **linear velocity** must be a tangent vector at the point.
  - We can relate this linear velocity to the **angular velocity**.

Imagine a point at a distance from the centre of rotation (we'll make the distance the radius just to make things easy). It has moved the arc length, ***s***, in a time interval.

- We can modify the classic **linear velocity** formula...

$$v=\frac{\Delta d}{\Delta t} \quad \text{becomes} \quad v=\frac{s}{\Delta t}$$

- From way back at the beginning of the notes, we defined the arc length in terms of the radius and radians, so that...

$$s = r\,\theta \quad \text{can be substituted in to get} \quad v=\frac{r\,\theta}{\Delta t} \quad \text{which we rewrite as} \quad v=r\frac{\Delta\theta}{\Delta t}$$

- But wait a minute, our original formula for **angular velocity** looked like that delta stuff...

$$\omega=\frac{\Delta\theta}{\Delta t} \quad \text{so sub out the delta stuff from} \quad v=r\frac{\Delta\theta}{\Delta t} \quad \text{and get...}$$

$$\boxed{v=r\,\omega}$$

v = linear velocity at a point (m/s)
r = radius (m)
ω = angular velocity (rad/s)

- So, for any point anywhere on the spinning object…
  - The **angular velocity** (**ω**) is a constant.
  - The **linear velocity** (**v**) is faster for any point further away from the centre of rotation.

Example 4: Looking back to **Example 2**, a 45 rpm record has a diameter of 17.5 cm and the label has a diameter of 3.82 cm. **Determine** the linear velocity of a point on the outer edge of record and of a point on the outer edge of the label.

Remember, the *angular* velocity is the same everywhere! We already calculated that in Example 2.
We need to find the radius of both the record and the label, then calculate the linear velocity.

Record
r = 17.5 cm ÷ 2 = 8.75 cm = 0.0875 m

$$v=r\,\omega$$
$$v=0.0875(4.71)$$
$$v=0.412125=0.41\,m/s$$

Label

r = 3.82 cm ÷ 2 = 1.91 cm = 0.0191 m

$$v = r\omega$$

$$v = 0.0191(4.71)$$

$$v = 0.089961 = 0.090\, m/s$$

Further away from the centre of rotation has a faster linear velocity.